A complementary group technique for a resolution of the outer multiplicity problem of $SU(n)$: (I) Littlewood rule and a complementary group of $SU(n)$

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Abstract

A complementary group to $SU(n)$ is found that realizes all features of the Littlewood rule for Kronecker products of $SU(n)$ representations. This is accomplished by considering a state of $SU(n)$ to be a special Gel'fand state of the complementary group $\mathcal{U}(2n-2)$. The labels of $\mathcal{U}(2n-2)$ can be used as the outer multiplicity labels needed to distinguish multiple occurrences of irreducible representations (irreps) in the $SU(n)\times SU(n)\downarrow SU(n)$ decomposition that is obtained from the Littlewood rule. Furthermore, this realization can be used to determine $SU(n)\supset SU(n-1)\times U(1)$ Reduced Wigner Coefficients (RWCs) and Clebsch-Gordan Coefficients (CGCs) of $SU(n)$, using algebraic or numeric methods, in either the canonical or a noncanonical basis. The method is recursive in that it uses simpler RWCs or CGCs with one symmetric irrep in conjunction with standard recoupling procedures. New explicit formulae for the multiplicity for $SU(3)$ and $SU(4)$ are used to illustrate the theory.

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I. Introduction

The Reduced Wigner Coefficients (RWCs) of $SU(n) \supset SU(n-1) \times U(1)$ are of importance in many physical applications. Except for those of $SU(2)$, which have been discussed extensively and expressed in various forms, RWCs of $SU(n) \supset SU(n-1) \times U(1)$, which can be used to evaluate CGCs of SU(n) in its canonical basis according to the Racah factorization lemma, have only been given analytically for some special cases. The biggest challenge involves the outer multiplicity in the decomposition of Kronecker products of $SU(n) \times SU(n) \downarrow SU(n)$. The first non-trivial but simplest $n = 3$ case was studied as part of the first applications of non-multiplicity-free CGCs of $SU(3)$ in nuclear and particle physics. There are several very distinct approaches to the problem: (i) a tensor operator method; (ii) an infinitesimal generator approach, in which matrix elements of $SU(n)$ generators are used to determine recursion relations for the RWCs and CGCs; (iii) a polynomial basis and generating invariants, in which a convenient model space is used to realize the basis of irreps; and (iv) use of the Schur-Weyl duality relation between $SU(n)$ and the symmetric group S_f. Among these are several ways of solving the problem; indeed, sometimes a combination of two or more methods is used. There are also different schemes for handling the outer multiplicity, especially for $SU(3)$, and these are usually referred to as either the canonical or a noncanonical labeling scheme.

A very thoroughly discussed approach to this problem is the canonical unit tensor operator method developed by Biedenharn and collaborators in a series of publications.[1−8] The unit tensor operator approach is particularly useful for deriving multiplicity-free CGCs of $U(n)$. The techniques that are part of this method have also proven to be useful in other approaches, but the method has not been used to produce a closed algebraic solution to the general outer multiplicity problem. This method was revisited in the late eighties in a Bargmann Hilbert space representation using the Vector Coherent State (VCS) theory.[9−11] Although the results seem no simpler than those found earlier, they do show that there is a relationship between $U(3) \supset U(2)$ RWCs and 3nj coefficients of $SU(2)$, with some of these being a consequence of the Schur-Weyl duality relation between the unitary and symmetric groups given by Ališuaskas et al.[12−14]

Noncanonical definitions of $SU(n)$ outer multiplicity labels, especially of $SU(3)$, have also been discussed rather extensively, for example by Moshinsky et al,[15−16] Derome and Sharp,[17−18] Resnikoff,[19] Pluhař et al.[20−21] A wider class of RWCs has been considered by Hecht,[22] Klimyk and Gavrilik,[23] and Le Blanc and Rowe,[24] who used definitions related to the canonical scheme. Generally, however, these results are for noncanonical labeling schemes. A further example is the extensive work of Ališauskas[25−28], who investigated paracanonical coupling relations and symmetries and various pseudo-canonical coupling schemes, which lead to biorthogonalities among the corresponding coefficients. It should be stated that noncanonical definitions for $SU(n)$ coupling coefficients normally lead to non-orthogonality with respect to the outer multiplicity. In such cases, the Gram-Schmidt process can be adopted to recover orthonormality, but this procedure includes an arbitrary choice in ordering the elements to be orthogonalized. Generally, only a

numerical algorithm is possible except a few simple cases where analytical expressions are available.[24−28]

The Schur-Weyl duality relation between $SU(n)$ and S_f was also used by several authors. It was first studied by Moshinsky,[16] Kramer,[29] and Alisauskas and Jucy,[14−16] who were able to demonstrate that the scheme works in the multiplicity-free and non-multiplicity-free cases. For non-multiplicity-free couplings, however, numerical orthogonalizition is required. This is illustrated for some simple cases in the work of Chen et al,[30] and by Pan and Chen for the $U_q(n)$ generalization of $U(n)$.[31]

Based on these methods, several packages have been developed for numerically evaluating CGCs of $U(n)$, especially of $SU(3)$. The earliest one is the well-known Akiyama-Draayer code for $SU(3)$ based on a combination of the tensor operator and infinitesimal generator methods.[32−33] Another is Chen's code for various couplings of $U(n)$ based on symmetric group techniques.[30] Still another is the RWC and CGC code for $SU(3)$ developed by Kaeding and Williams.[34−36]

Very recently, Parkash and Sharatchandra worked out an algebraic formula for the general CGCs of $SU(3)$.[37] The method used in their paper is based on a polynomial realization in Bargmann space using generating functions, which was first studied by Shelepin and Karasev for the multiplicity-free case.[38−39] The final results are expressed in terms of a restricted sum over 33 variables up to a normalization factor. To determine the value of a single CGC within this formulation is not easy; neither the algebraic nor numerical results are simple. Nevertheless, it is the first algebraic expression for CGCs of $SU(3)$ with multiplicity. It should be noted, however, that to extend this method to $n \geq 4$ cases will be much more complicated. Therefore, another simpler and more direct approach to a resolution of the outer multiplicity problem for $SU(n)$ is necessary.

The present paper is the first (I) in a series which has this as its objective. First of all, a complementary group $\mathcal{U}(2n-2)$ realization of the Kronecker product $SU(n) \times SU(n) \downarrow SU(n)$ is found according to the well-known Littlewood rule. The scheme gives a simple resolution of the outer multiplicity. An analysis of the Littlewood rule is also used to derive a new multiplicity formulae for $SU(n)$. Examples are given for the $SU(3)$ and $SU(4)$ cases which can, in principle, be extended to $SU(n)$. A procedure for evaluating CGCs or RWCs of $SU(n) \supset SU(n-1) \times U(1)$ is outlined which uses recoupling procedures. By using this method, $SU(n) \supset SU(n-1) \times U(1)$ RWCs or CGCs with outer multiplicity can be obtained analytically in some simple cases or numerically in general. Detailed results will be given for the $SU(3)$ and $SU(4)$ cases in Parts II and III of the series, respectively. It should be noted that the RWCs or CGCs obtained in this way are orthogonal with respect to the outer multiplicity labels and therefore the scheme that is canonical.

II. Littlewood rule and the complementary group

The Littlewood rule for determining Kronecker products of $SU(n)$ in $SU(n)\times SU(n)\downarrow SU(n)$ is a reflection of the Shcur-Weyl duality relation between $SU(n)$ and the symmetric group S_f. According to Schur-Weyl duality relation, an irrep $[\lambda]$ of $SU(n)$ can also be regarded as the same irrep of S_f with $\sum_{i=1}^{n}\lambda_i = f$. Therefore, the Kronecker product of two $SU(n)$ irreps $[\lambda]\times[\mu]$ in the decomposition $SU(n)\times SU(n)\downarrow SU(n)$ can be obtained from the product of two S-functions of the corresponding symmetric groups:

$$[\lambda]\times[\mu] = \sum_{\nu}\{\lambda\mu\nu\}[\nu], \tag{2.1}$$

where $\{\lambda\mu\nu\}$ is the number of occurrence of $[\lambda]$ in the product. To determine all the irreps that appear on the rhs of (2.1), one can use the well-known Littlewood rule:[40] First fill in the Young diagram $[\mu]=[\mu_1,\mu_2,\cdots,\mu_n]$ with μ_1 symbols a_1 in the first row, μ_2 symbols a_2 in the second row, μ_3 symbols a_3 in the third row, $\cdots$, and μ_n symbols a_n in the nth row. Then, the final irrep denoted by Young diagram $[\nu]$ can be obtained by augmenting the Young diagram $[\lambda]$ with the μ_1 a_1 symbols, μ_2 a_2 symbols,$\cdots$, and μ_n a_n symbols, respectively, in ways specified by the following three conditions:

(a) No identical symbols should appear in the same column of the diagram.

(b) If the a_1, a_2, $\cdots$, a_n symbols are counted from right to left starting at the top, then at each stage the number of a_1 symbols must not be less than the number of a_2 symbols, which must not be less than the number of a_3 symbols, and so on.

(c) The Young diagram $[\nu]$ obtained after the addition of each symbol must be standard, that is, $\nu_1\geq\nu_2\geq\cdots\geq\nu_n$.

The Young diagram $[\nu]$ filled with symbols a_1, a_2, $\cdots$, a_n under restrictions (a)–(c) can be regarded as a special Weyl tableau of a unitary group. Recall some basic definitions for Weyl tableau: A Weyl tableau is a Young diagram with the boxes filled by a set of ordered indices a_1, a_2, $\cdots$, a_n. The filling must be done such that:

(i) no identical symbols should appear in the same column,

(ii) the symbols must be in nondecreasing order from left to right in any row and in increasing order from top to bottom in any column.

The one-to-one correspondence between the Gel'fand symbol and the Weyl tableau is realized in the following way:

$$\left(\begin{array}{c} [\nu] \\ (m) \end{array} \right) = W^{[\nu]} = \begin{array}{|l|l|l|l|l|} \hline f_{11}a_1\text{'s} & f_{12}a_2\text{'s} & \cdots & \cdots & f_{1n}a_n\text{'s} \\ \hline f_{22}a_2\text{'s} & f_{23}a_3\text{'s} & \cdots & f_{2n}a_n\text{'s} \\ \cline{1-4} \multicolumn{3}{|l|}{\cdots\cdots\cdots\cdots} \\ \cline{1-3} f_{nn}a_n\text{'s} \\ \cline{1-1} \end{array} \tag{2.2}$$

where

$$f_{1k} = m_{1k} - m_{1k-1}, \quad f_{2k} = m_{2k} - m_{2k-1}, \cdots,$$

$$f_{k-1k} = m_{k-1k} - m_{k-1k-1}, \quad f_{kk} = m_{kk}. \tag{2.3}$$

In other words, a Weyl tableau $W^{[\nu]}$ filled with a_1, $a_2, \cdots$, a_n, corresponds to the n partitions, $[\nu](= [m_{in}])$, $[m_{in-1}], \cdots$, $[m_{i2}]$ and $[m_{11}]$ of a Gel'fand symbol, where $[m_{ik}]$ is the Young diagram resulting from deleting all the boxes in the Weyl tableau occupied by the symbols $a_n, \ a_{n-1}, \ \cdots, \ a_{k+1}$.

It is clear that the definitions of the Weyl tableau and the rules for placing symbols in a Young diagram given by Littlewood are the same eccept for some of the restrictions given by (b). that is, a_k's can appear in the ith rows with $i < k$, and the number of a_k's can be greater than that of a_i's with $i < k$ from right to left and from top to bottom, while these cases are forbidden by the restriction (b) of the Littlewood rule. Therefore, it is obvious that the Littlewood rule for placing symbols in a Young diagram can be regarded as a special Weyl tablueau for a unitary group. Hence, under the restrictions of Littlewood rule given by (b), one obtains a special Gel'fand basis of a corresponding unitary group, which is called the complementary group for Kronecker products of $SU(n)$.

Assume the irrep $[\lambda]$ has p_1 rows, while $[\mu]$ has p_2 rows. Then, the final irrep $[\nu]$ has at most $p_1 + p_2$ rows with $p_1 + p_2 \leq n$. Therefore, the complementary group corresponding to the Kronecker product of $SU(n)$ is $\mathcal{U}(p_1 + p_2)$. A general $SU(n)$ irrep has at most $n - 1$ rows because one can always use the equivalence condition $[m_{1n}m_{2n} \cdots m_{nn}] = [m_{1n} - m_{nn}, m_{2n} - m_{nn}, \cdots, m_{1n-1} - m_{nn}]$ to remove the nth row if it exists. From this it follows that the minimum complementary group is $\mathcal{U}(2n - 2)$ for general Kronecker products of $SU(n)$.

Using the correspondence between Weyl tableau and a Gel'fand symbol, one can easily find the following relations among coupled and uncoupled state labels of $\mathcal{U}(2n - 2)$.

$$\begin{array}{|c|} \hline \quad\quad\lambda_1\quad\quad \\ \hline \lambda_2 \\ \hline \cdots\cdots \\ \hline \lambda_{n-1}\text{'s} \\ \hline \end{array} \Rightarrow \left(\begin{array}{c} [\lambda_1\lambda_2\cdots\lambda_{n-1}\dot{0}] \ \ \mathcal{U}(2n-2) \\ \cdots\cdots \\ [\lambda_1\lambda_2\cdots\lambda_{n-1}\dot{0}] \ \ \mathcal{U}(n-1) \\ \rho \end{array} \right) \tag{2.4a}$$

$$\begin{array}{|l|}\hline \qquad\mu_1\ a_1\text{'s}\qquad\qquad \\ \hline \quad\mu_2\ a_2\text{'s}\quad \\ \hline \quad\cdots\cdots \\ \hline \mu_{n-1}\ a_n\text{'s} \\ \hline \end{array} \Rightarrow \begin{pmatrix} [\mu_1\mu_2\cdots\mu_{n-1}\dot{0}] & \mathcal{U}(2n-2) \\ [\mu_1\mu_2\cdots\mu_{n-2}\dot{0}] & \mathcal{U}(2n-3) \\ \cdots\cdots & \\ [\mu_1\mu_2\dot{0}] & \mathcal{U}(n+1) \\ [\mu_1\dot{0}] & \mathcal{U}(n) \\ [\dot{0}] & \mathcal{U}(n-1) \end{pmatrix} \tag{2.4b}$$

while the final coupled $\mathcal{U}(2n-2)$ basis is

$$\begin{pmatrix} [\nu_1\nu_2\cdots\nu_n\dot{0}]\ (\tau) & \mathcal{U}(2n-2) \\ \\ (\tau) & \\ \\ [\lambda_1\lambda_2\cdots\lambda_{n-1}\dot{0}] & \mathcal{U}(n-1) \\ \rho & \end{pmatrix}, \tag{2.4c}$$

where (τ) stands for intermediate sublabels between $\mathcal{U}(2n-2)$ and $\mathcal{U}(n-1)$ given by the Littlewood rule, which is simultaneously the outer multiplicity label of both $\mathcal{U}(2n-2)$ and $SU(n)$, and ρ represents sublabels of $\mathcal{U}(n-1)$.

Therefore, (τ) can be regarded as multiplicity labels of $SU(n)$. For example, the final coupled state can be written as

$$\begin{pmatrix} [\nu_1\nu_2\cdots\nu_n]\ (\tau) \\ (\nu) \end{pmatrix}, \tag{2.5}$$

where (ν) stands for sublabels of $SU(n)$. Expression (2.5) is similar to the upper Gel'fand pattern introduced by Biedenharn et al.[1−8] The final coupled $\mathcal{U}(2n-2)$ labels (τ) in (2.4c) provide the outer multiplicity labels needed in the decomposition $[\lambda] \times [\mu] \downarrow [\nu]$. This will be discussed further in the next section.

III. Outer multiplicity problem of SU(3) and SU(4)

As noted above, the outer multiplicity in the decomposition of the Kronecker products of $SU(n) \times SU(n) \downarrow SU(n)$ is the main obstacle in applications of algebraic methods to physical problems. There are a lot of articles devoted to this subject. In order to resolve the problem for the $SU(3)$, Hecht[22] proposed an external labeling operator of third order, an operator that may be related to the one proposed by Moshinsky[16] in terms of the complementary $U(4) \supset U(2) \times U(2)$ chain. Alisauskas and Kulish[41] have also proposed an external labeling operator, a fourth order form suggested by Sharp[42] in a study of Yang-Baxter equations. There are also other articles on this subject. For example, new Casimir operators, the so called chiral Casimirs, were introduced in [16,

43-44]. Also, various formulae[19,45−48] for the multiplicity of $SU(3)$ exist in the literature, however, such expressions are normally not linked to the $SU(3)$ coupling and recoupling coefficients problem. There is still no general formula for the outer multiplicity of $SU(n)$ with $n \geq 4$. In this article and forthcoming papers, the complementary group $\mathcal{U}(2n-2)$ to the $SU(n) \times SU(n) \downarrow SU(n)$ will be shown to be a powerful tool for deriving both multiplicity formulae and coupling and recoupling coefficients of $SU(n)$. Multiplicity formulae for $SU(3)$ and $SU(4)$ are considered below.

(1) SU(3) case. Consider the general Keronecker product $(\lambda_1\mu_1) \times (\lambda_2\mu_2)$, where the well-known notation for $SU(3)$ in physics is adopted. The irrep $(\lambda\mu)$ can be expressed in terms of a two-rowed Young diagram $[\nu_1\nu_2]$ with $\nu_1 = \lambda + \mu$, and $\nu_2 = \mu$. Using the Littlewood rule, the decomposition of $(\lambda_1\mu_1) \times (\lambda_2\mu_2)$ can be expressed in terms of a quintuple sum.

$$(\lambda_1\mu_1) \times (\lambda_2\mu_2) = \sum_{k_1=0}^{\lambda_2+\mu_2} \sum_{k_2=0}^{\min(\lambda_1,\ \lambda_2+\mu_2-k_1)} \sum_{k_3=0}^{\min(\mu_1,\ \lambda_2+\mu_2-k_1-k_2)} \sum_{n_1=0}^{\min(\lambda_1+k_1-k_2,\ \mu_2,\ k_1)} \times$$

$$\sum_{n_2=0}^{\min(\mu_2-n_1,\ \mu_1+k_2-k_3,\ k_1+k_2-n_1)} [\lambda_1+\mu_1+k_1,\ \mu_1+k_2+n_1,\ k_3+n_2], \tag{3.1}$$

where the constraints

$$\sum_{i=1}^{3} k_i = \lambda_2 + \mu_2, \quad \sum_{i=1}^{2} n_i = \mu_2 \tag{3.2}$$

apply in the summation. Expression (3.1) can be further simplified, for example, to O'Reilly's formula[47] in which only a triple sum appears. However, (3.1) can be used to help determine a multiplicity formula and determine the multiplicity labels of the complementary group.

Consider a Young diagram of the resultant irrep $[m_1m_2m_3]$ according to (3.1) with conditions given by (3.2):

$$\begin{array}{l} \boxed{\qquad\qquad \lambda_1+\mu_1 \qquad\qquad}\boxed{\quad k_1 \quad {}^{\alpha}} \\ \boxed{\qquad \mu_1 \qquad}\boxed{m_2-\mu_1-\eta \ \ {}^{\alpha}}\boxed{\ \eta \ \ {}^{\beta}} \\ \boxed{m_3-\mu_2+\eta \ \ {}^{\alpha}}\boxed{\mu_2-\eta \ \ {}^{\beta}} \end{array} \tag{3.3}$$

where α and β are the a_i symbols of the Littlewood rule for $SU(3)$. The labels in (3.3) have been arranged to acommodate the constraints of (3.2) and to yield a multiplicity formula very easily. In this forms it is obvious that a diagram with the same number of

boxes in each row can only appear repeatedly when η is not a fixed integer. Therefore, η can be regarded as the multiplicity label of SU(3).

According to Littlewood rule (a)–(c), it is easy to derive the following limits on η:

$$\eta_{\min} \leq \eta \leq \eta_{\max}, \tag{3.4}$$

where

$$\eta_{\max} = \min(m_1 - \lambda_1 - \mu_1,\ \mu_2,\ m_2 - \mu_1,\ \lambda_2 + \mu_2 - m_3,\ \mu_1 + \mu_2 - m_3,\ m_2 - m_3),$$
$$\eta_{\min} = \max(0,\ \mu_2 - m_3,\ m_2 - \lambda_1 - \mu_1). \tag{3.5}$$

Hence, the multiplicity of $[m_1 m_2 m_3] \equiv (m_1 - m_2,\ m_2 - m_3)$ occurring in the Kronecker product $(\lambda_1 \mu_1) \times (\lambda_2 \mu_2)$ is given by

$$\mathrm{Multi}(SU_3) = \eta_{\max} - \eta_{\min} + 1. \tag{3.6}$$

This expression is very simple and more transparent than others found in the literature.

In this case, the complementary group is $\mathcal{U}(4)$. The Gel'fand symbol of $\mathcal{U}(4)$ corresponding to the resultant irrep of $SU(3)$ given in (3.3) is

$$\begin{pmatrix} [m_1 m_2 m_3 0]\ \eta \\ [m_1\ m_2 - \eta\ \ m_3 - \mu_2 + \eta] \\ [\lambda_1 + \mu_1\ \mu_1] \\ \rho \end{pmatrix},\ \eta = \eta_{\min},\ \eta_{\min} + 1, \cdots,\ \eta_{\max}, \tag{3.7}$$

where ρ is the intrinsic label for $\mathcal{U}(1)$, which is not important for our purpose. Some conditions in (3.5) can also be easily obtained from the betweeness conditions of the entries in the Gel'fand symbol (3.7). However, the remaining conditions in (3.5) can only be deduced from the Littlewood rule (b), and can not be obtained from the betweeness conditions. Hence, only one outer multiplicity label is needed in the decomposition of $SU(3) \times SU(3) \downarrow SU(3)$. This is why the CGCs of $SU(3)$ can be determined numerically by using only one type of tensor operator.[32–36]

In contrast with the so-called canonical labeling scheme proposed by Biedenharn et al., in which three independent shifts determined by an upper pattern are introduced, the complementary $\mathcal{U}(4)$ group provides only one outer multiplicity label in the $SU(3)$ case. The complementary group labeling scheme is therefore a very economical way to label the outer multiplicity of $SU(3)$, and by extension, of $SU(n)$.

Furthermore, the upper pattern labeling scheme given by Biedenharn et al is also equivalent to our labeling scheme, which will be proved in our next paper. However, similar to the complementary group labeling scheme, some restrictions on the ranges of Γ_{22} in the upper pattern $[\Gamma_{12},\ \Gamma_{22}] = [m_1 + m_2 - \lambda_1 - 2\mu_1 - \Gamma_{22},\ \Gamma_{22}]$ for the coupling $(\lambda_1\mu_1) \times (\lambda_2\mu_2) \downarrow [m_1m_2m_3]$ should be obtained from the Littlewood rule of $SU(3)$. Actually, the ranges of Γ_{22} should be the same as those of η given by (3.5), which, however, can not be derived directly from restrictions on upper pattern labels. For example, [422] occurs only once in the decomposition $[310] \times [310]$. However, there are two sets of upper pattern labels

$$\begin{pmatrix} & \Gamma_{11} & \\ \Gamma_{12} & & \Gamma_{22} \end{pmatrix} = \begin{pmatrix} & 1 & \\ 2 & & 0 \end{pmatrix} \ ; \ \begin{pmatrix} & 1 & \\ 1 & & 1 \end{pmatrix}$$

are allowed according to the upper pattern labeling scheme. Actually, the state labelled by $\begin{pmatrix} & 1 & \\ 1 & & 1 \end{pmatrix}$ should be eleminated according to the Littlewood rule. Therefore, restrictions from Littlewood rule must apply to the upper pattern labeling scheme, which was not mentioned in their papers [1-8], and indeed difficult to be obtained directly from their methods.

(2) SU(4) case. A general $SU(4)$ irrep has three rows. Using the Littlewood rule, the following formula for the decomposition of $SU(4) \times SU(4) \downarrow SU(4)$ can be determined:

$$[\lambda_1\lambda_2\lambda_3] \times [\mu_1\mu_2\mu_3] = \sum_{k_1=0}^{\mu_1} \sum_{k_2=0}^{\min(\mu_1-k_1,\lambda_1-\lambda_2)} \sum_{k_3=0}^{\min(\lambda_2-\lambda_3,\mu_1-k_1-k_2)} \sum_{k_4=0}^{\min(\lambda_3,\mu_1-k_1-k_2-k_3)} \times$$
$$\sum_{l_1=0}^{\min(\lambda_1+k_1-\lambda_2-k_2,\mu_2,k_1)} \sum_{l_2=0}^{\min(\lambda_2+k_2-\lambda_3-k_3,\mu_2-l_1,k_1+k_2-l_1)} \times$$
$$\sum_{l_3=0}^{\min(\lambda_3+k_3-k_4,\mu_2-l_1-l_2,k_1+k_2+k_3-l_1-l_2)} \sum_{n_1=0}^{\min(\lambda_2+k_2+l_1-\lambda_3-k_3-l_2,\mu_3,l_1)} \times$$
$$\sum_{n_2=0}^{\min(\lambda_3+k_3+l_2-k_4-l_3,\mu_3-n_1,l_1+l_2-n_1)} [\lambda_1+k_1,\ \lambda_2+k_2+l_1,\ \lambda_3+k_3+l_2+n_1,\ k_4+l_3+n_2], \quad (3.8)$$

where the following constraints

$$\sum_{i=1}^{4} k_i = \mu_1, \quad \sum_{i=1}^{3} l_1 = \mu_2, \quad \sum_{i=1}^{2} n_i = \mu_3 \qquad (3.9)$$

apply in the summation. In the resultant irrep $[\nu_1\nu_2\nu_3\nu_4] \equiv [\lambda_1 + k_1,\ \lambda_2 + k_2 + l_1,\ \lambda_3 + k_3 + l_2 + n_1,\ k_4 + l_3 + n_2]$ with the restrictions given by (3.9) there may be six ways to relabel the configuration which leave the irrep unchanged:

$$\begin{array}{|l|l|l|l|}
\hline
\lambda_1 & k_1 \ {}^{\alpha} & & \\
\hline
\lambda_2 & k_2 \ {}^{\alpha} & l_1 \ {}^{\beta} & \\
\hline
\lambda_3 & k_3 \ {}^{\alpha} & l_2 \ {}^{\beta} & n_1 \ {}^{\gamma} \\
\hline
k_4 \ {}^{\alpha} & l_3 \ {}^{\beta} & n_2 \ {}^{\gamma} & \\
\hline
\end{array} \qquad (3.10a)$$

where

$$k_1 = \nu_1 - \lambda_1,$$

$$k_2 = \nu_2 - \lambda_2 - \xi_1 - \xi_2,$$

$$k_3 = \nu_3 - \lambda_3 - \mu_2 + \xi_1 - \xi_3 - \xi_5 - \xi_6,$$

$$k_4 = \nu_4 - \mu_3 + \xi_2 + \xi_3 + \xi_5 + \xi_6,$$

$$l_1 = \xi_1 + \xi_2,$$

$$l_2 = \mu_2 - \xi_1 - \xi_4 + \xi_5,$$

$$l_3 = \xi_4 - \xi_2 - \xi_5,$$

$$n_1 = \xi_3 + \xi_4 + \xi_6,$$

$$n_2 = \mu_3 - \xi_3 - \xi_4 - \xi_6, \qquad (3.10b)$$

and α, β, and γ are the symbols filling in each box according to the Littlewood rule. However, by using the following transformation

$$\eta_1 = \xi_1 + \xi_2, \quad \eta_2 = \xi_3 + \xi_4 + \xi_6, \quad \eta_3 = \xi_4 - \xi_2 - \xi_5, \qquad (3.11)$$

it can be shown that only three variables η_i with $i = 1,\ 2$, and 3 are independent. Therefore, (3.10) can be relabelled in terms of these 3 variables,

$$k_1 = \nu_1 - \lambda_1,$$

$$k_2 = \nu_2 - \lambda_2 - \eta_1,$$

$$k_3 = \nu_3 - \lambda_3 - \mu_2 + \eta_1 - \eta_2 + \eta_3,$$

$$k_4 = \nu_4 - \mu_3 + \eta_2 - \eta_3,$$

$$l_1 = \eta_1,$$

$$l_2 = \mu_2 - \eta_1 - \eta_3,$$

$$l_3 = \eta_3,$$

$$n_1 = \eta_2,$$

$$n_2 = \mu_3 - \eta_2. \tag{3.12}$$

Applying the Littlewood rule to this result yields the following boundary conditions for the outer multiplicity labels η_1, η_2, and η_3.

$$\eta_{1\,\min} \le \eta_1 \le \eta_{1\,\max}, \quad \eta_{2\,\min} \le \eta_2 \le \eta_{2\,\max}, \quad \eta_{3\,\min} \le \eta_3 \le \eta_{3\,\max}, \tag{3.13}$$

where

$$\eta_{1\,\min} = \max(0,\ \nu_2 - \lambda_1), \quad \eta_{1\,\max} = \min(\nu_2 - \lambda_2,\ \nu_1 - \lambda_1),$$

$$\eta_{2\,\min} = \max(\nu_3 - \nu_2 + \eta_1,\ 0), \quad \eta_{2\,\max} = \min(\eta_1,\ \mu_3,\ \nu_3 - \nu_4),$$

$$\eta_{3\,\min} = \max(2\eta_2 - \eta_1 + \mu_2 + \nu_4 - \nu_3 - \mu_3,\ \eta_2 - \eta_1 + \lambda_3 + \mu_2 - \nu_3, \eta_2 + \nu_4 - \lambda_3 - \mu_3,$$

$$\eta_2 + \lambda_1 + \lambda_2 + \lambda_3 + 2\mu_2 - \nu_1 - \nu_2 - \nu_3,\ \eta_1 + \lambda_1 + \lambda_2 + \mu_2 - \nu_1 - \nu_2,$$

$$0,\ \mathrm{Int}[(\eta_2 + \lambda_1 + \lambda_2 + \lambda_3 + 2\mu_2 - \nu_1 - \nu_2 - \nu_3)/2]),$$

$$\eta_{3\,\max} = \min(\mu_2 - \eta_1,\ \nu_4 - \mu_3 + \eta_2,\ \mu_2 - \mu_3,\ \lambda_2 - \nu_3 + \mu_2 - \eta_1 + \eta_2,\ \mu_2 - \eta_2), \tag{3.14}$$

where Int$[x]$ is the integer part of x. Thus, the multiplicity of $[\nu_1,\ \nu_2,\ \nu_3,\ \nu_4] \equiv [\nu_1 - \nu_4,\ \nu_2 - \nu_4,\ \nu_3 - \nu_4]$ appearing in the Kronecker product $[\lambda_1\lambda_2\lambda_3] \times [\mu_1\mu_2\mu_3]$ can be calculated by

$$\mathrm{Multi}(SU_4) = \sum_{\eta_1=\eta_{1\,\min}}^{\eta_{1\,\max}} \ \sum_{\eta_2=\eta_{2\,\min}(\eta_1)}^{\eta_{2\,\max}(\eta_1)} \ \sum_{\eta_3=\eta_{3\,\min}(\eta_1,\ \eta_2)}^{\eta_{3\,\max}(\eta_1,\ \eta_2)} . \tag{3.15}$$

The complementary group of the Kronecker product $[\lambda_1\lambda_2\lambda_3] \times [\mu_1\mu_2\mu_3]$ of $SU(4)$ is $\mathcal{U}(6)$ with the following special Gel'fand labels

$$\begin{pmatrix} [\nu_1\ \nu_2\ \nu_3\ \nu_4]\ (\eta_1\eta_2\eta_3) & \mathcal{U}(6) \\ [\nu_1,\ \nu_2,\ \nu_3 - \eta_2,\ \nu_4 - \mu_3 + \eta_2,\ 0] & \mathcal{U}(5) \\ [\nu_1,\ \nu_2 - \eta_1,\ \nu_3 - \mu_2 + \eta_1 + \eta_3 - \eta_2,\ \nu_4 - \mu_3 + \eta_2 - \eta_3] & \mathcal{U}(4) \\ [\lambda_1\ \lambda_2\ \lambda_3] & \mathcal{U}(3) \\ \rho & \end{pmatrix}. \tag{3.16}$$

The Gel'fand labels of $[\lambda_1\lambda_2\lambda_3\dot{0}]$ and $[\mu_1\mu_2\mu_3\dot{0}]$ for $\mathcal{U}(6)$ are

$$\begin{pmatrix} [\lambda_1\lambda_2\lambda_3\dot{0}] & \mathcal{U}(6) \\ [\lambda_1\lambda_2\lambda_3\dot{0}] & \mathcal{U}(5) \\ [\lambda_1\lambda_2\lambda_3 0] & \mathcal{U}(4) \\ [\lambda_1\lambda_2\lambda_3] & \mathcal{U}(3) \\ \rho & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} [\mu_1\mu_2\mu_3\dot{0}] & \mathcal{U}(6) \\ [\mu_1\mu_2\dot{0}] & \mathcal{U}(5) \\ [\mu_1 0] & \mathcal{U}(4) \\ [\dot{0}] & \mathcal{U}(3) \end{pmatrix}. \tag{3.17}$$

Again, most of the boundary conditions for the multiplicity labels η_1, η_2, and η_3 can be obtained from the betweeness conditions for the Gel'fand symbol shown in (3.16). However, the remaining conditions can only be deduced from the Littlewood rule because (3.16) is a special Gel'fand basis for the canonical chain $\mathcal{U}(6) \supset \mathcal{U}(5) \supset \cdots \supset \mathcal{U}(2) \supset \mathcal{U}(1)$. From this development it is clear that there are at most 3 quantum numbers needed to label the outer multiplicity for the decomposition $SU(4) \times SU(4) \downarrow SU(4)$. In the canonical unit tensor approach proposed by Biedenharn et al. for the $SU(4)$ case, there are 4 shifts out of 6 upper labels, of which only 3 labels are independent.[1–8] Similar to $SU(3)$ case, restrictions from Littlewood rule must apply to eliminate superfluous multiplicity states in the upper pattern labeling scheme.

It should be noted that any $SU(n)$ function, for example, CGCs, RWCs, or Racah coefficients, etc., is rank n independent, and only depends on boxes contained in the Young diagrams of the corresponding irreps because of the Schur-Weyl duality relation between $SU(n)$ and S_f. For example, the multiplicity of Kronecker product for two two-rowed irreps of $SU(n)$ is the same as that of $SU(3)$, and that for two three-rowed irreps is the same as that of $SU(4)$, and so on. Hence, the results for $SU(3)$ and $SU(4)$ apply

for the general $SU(n)$ case as well. As a trivial example, note that the $SU(3)$ multiplicity expression follows from the one for $SU(4)$ in the two-rowed limit ($\eta_1 \rightarrow \eta$, $\eta_2 = 0$ and $\eta_3 = 0$) of the theory though this fact can not be clearly seen from (3.14).

IV. Conclusions

In this paper, a complementary group $U(2n-2)$ to $SU(n)$ is found that gives a complete realization of all the features of the Littlewood rule in the Kronecker product decomposition of $SU(n) \times SU(n) \downarrow SU(n)$. By using this scheme, the outer multiplicity labels for $SU(n)$ can be easily assigned, being nothing other than a set of sublabels of the special Gel'fand basis of the complementary $\mathcal{U}(2n-2)$ group. Furthermore, within this framework, most of the boundary conditions on the multiplicity labels can be easily obtained from the betweenness conditions of the Gel'fand symbols of $\mathcal{U}(2n-2)$, while the remaining conditions must be deduced from the Littlewood rule. The method was used to obtained simple multiplicity formulae for $SU(3)$ and $SU(4)$. In addition, in the coupling of two $SU(n)$ irreps, the basis for $SU(n)$ can further be labeled by the final $\mathcal{U}(2n-2)$ sublabels η_i obtained from the coupling of two uncoupled basis vectors of the corresponding specail Gel'fand basis of $\mathcal{U}(2n-2)$, which are missing within the $SU(n)$ group. This situation is very similar to that of the canonical unit tensor approach proposed by Biedenharn et al. However, in the canonical unit tensor approach, there are n independent shifts indicated by the upper pattern of $U(n)$ from the $n(n-1)/2$ upper labels. While these upper indices can be used to label the outer multiplicity of $U(n)$, there may very well be superfluous degree-of-freedom among the labels and these may be eliminated, especially, restrictions from Littlewood rule of $SU(n)$ must apply to eliminate superfulous multiplicity states which are not allowed in the decomposition.

It should be stated that the same complementary group to the resolution of $SU(n)$ was also considered in [15]. However, the method used and the final outcome are all different. Firstly, In [15], this complementary group was derived by using boson realizations given by Moshinsky.[16] While it now comes naturally from the Littlewood rule. Secondly, according to [15], the complementary group should be labeled in terms of a noncanonical chain $\mathcal{U}(2n-2) \supset \mathcal{U}(n-1) \times \mathcal{U}(n-1)$. In this way, the RWCs of $SU(n)$ still can not easily be derived because new inner multiplicity occurs in the decomposition $\mathcal{U}(2n-2) \downarrow \mathcal{U}(n-1) \times \mathcal{U}(n-1)$. In order to overcome this difficulty, another kind of Wigner coefficients, the so called auxiliary Wigner coefficients was defined in [15], which is different from the standard definition of WCs, and satisfy another type of orthogonality conditions. We shall discuss these special WCs in the next paper. It shall be show in the next paper that one can derive analytical expressions in some simple cases and the corresponding algorithms for $SU(n)$ RWCs or CGCs with multiplicity in general in both the canonical and noncanonical bases within this labeling scheme if the multiplicity-free coefficients in these bases are known.

To reiterate an important point, the complementary group $\mathcal{U}(2n-2)$ scheme for labeling outer multiplicities in Kronecker products of $SU(n)$ is itself a canonical scheme

because the basis of $SU(n)$ labeled in this way is orthogonal with respect to the outer multiplicity labels. A general procedure for evaluating RWCs or CGCs for $SU(3)$ and $SU(4)$ will be given in the forthcoming papers.

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